

Lattice gauge theories and the Florentino conjecture

Andrei Tyurin

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Abstract

We study the relation between the space of representation classes of the fundamental group of a Riemann surface and gauge theory on trivalent graphs. We construct a partial gauge fixing in the latter gauge theory. As an application we get a proof of a conjecture of Florentino.

1 Introduction

Two boundary cases of gauge theories are extremely useful for explorations: gauge theory with a discrete gauge group (see Freed and Quinn [2]) and gauge theory over a discrete space, usually called a “lattice model”. Pragmatically, such lattice models are required as a test of the concepts. In this paper we explore from a “theoretical” point of view the second case: gauge theory on graphs, more precisely, on trivalent graphs.

At the end of [1, Seminar 6], Michael Atiyah proposed a programme to make sense of the functional integral of Chern–Simons theory for finite level k in a rigorous way. Because direct attempt at the analysis is extremely difficult he proposed to think of connections in purely combinatorial terms. In particular using a discrete analogue of gauge theory, one should express the Chern–Simons functional in a combinatorial framework. For this, we investigated geometric properties of the phase map in [3]. On the other hand, we extended to spin networks all the constructions of TQFT concerning Wilson loops (knots). This made possible a partial realization of Atiyah’s programme using trivalent graphs instead of loops.

Unfortunately we don't have enough space to recall all the details of this treatment. See [4] and [5], from which the current paper derives.

Our starting point is of course exactly the same as in a number of texts about CFT, quantum gravity and spin networks. Our previous paper [6] contains a brief survey of mathematical aspects of $SU(2)$ -spin networks of genus g . Here we extend the study of the “pumping trick” geometry to obtain a close relation between gauge theory on graphs and gauge theory on Riemann surfaces. The construction and results can be summarized as follow. Any graph Γ defines a handlebody $\tilde{\Gamma}$, that is, a 3-manifold with boundary a Riemann surface $\partial\tilde{\Gamma} = \Sigma_\Gamma$ by pumping up the edges of Γ to tubes and the vertices to small 2-spheres with 3 holes. To investigate the relation between the space of classes of $SU(2)$ -representations of the fundamental group of the Riemann surface $\partial\tilde{\Gamma} = \Sigma_\Gamma$ and the $SU(2)$ -gauge theory on Γ we construct the partial gauge fixing in the latter gauge theory. As an application of this gauge fixing we get the proof of the Florentino conjecture.

To be concrete we only consider trivalent graphs and start by recalling some foundations of graph geometry in a form that is convenient for us.

2 Geometry of trivalent graphs

Write $E(\Gamma)$ and $V(\Gamma)$ for the set of edges and vertices of a graph Γ , and $E(\Gamma)_v$ for the set of edges out of v . Write $F(\Gamma) = \{v \in e\}$ and $L(\Gamma)$ for the set of flags and loops of Γ , where a flag is an edge with a fixed end, and a loop is an edge with only one vertex. The latter set is a subset $L(\Gamma) \subset E(\Gamma)$ and sending a loop to its vertex gives the map $v: L(\Gamma) \rightarrow V(\Gamma)$ which is an embedding:

$$L(\Gamma) = v(L(\Gamma)) \subset V(\Gamma) \quad (2.1)$$

because we only consider trivalent graphs.

The two projections $e: F(\Gamma) \rightarrow E(\Gamma)$ and $v: F(\Gamma) \rightarrow V(\Gamma)$ are ramified covers of degree 2 and 3 having the same ramification locus $W_e = W_v \subset F(\Gamma)$, consisting of pairs $v \in e$ where the edge e is a loop. The branch loci of these covers are

$$R_e = L(\Gamma) \subset E(\Gamma) \quad \text{and} \quad R_v = v(L(\Gamma)) \subset V(\Gamma). \quad (2.2)$$

Write $|\cdot|$ for the number of elements of a finite set. Then

$$2 \cdot |E(\Gamma)| - |L(\Gamma)| = |F(\Gamma)| = 3 \cdot |V(\Gamma)| - |L(\Gamma)|. \quad (2.3)$$

Hence $|V(\Gamma)| = 2g - 2$ and $|E(\Gamma)| = 3g - 3$, where $g > 1$ is a certain integer called the genus of Γ .

A path of length 1 in Γ is just an oriented edge \vec{e} . Let $P_1(\Gamma)$ be the set of paths of length 1 in Γ . Reversing the orientation of an edge e defines an involution

$$i_e: P_1(\Gamma) \rightarrow P_1(\Gamma). \quad (2.4)$$

These edgewise involutions generate a group $O(\Gamma) = \mathbb{Z}_2^{3g-3}$ acting on $P_1(\Gamma)$, with quotient

$$P_1(\Gamma)/O(\Gamma) = E(\Gamma), \quad (2.5)$$

and the corresponding map

$$O: P_1(\Gamma) \rightarrow E(\Gamma). \quad (2.6)$$

Any section

$$o: E(\Gamma) \rightarrow P_1(\Gamma) \quad (2.7)$$

of this map is called an orientation of Γ . The group $O(\Gamma)$ contains a diagonal involution

$$i_\Delta: P_1(\Gamma) \rightarrow P_1(\Gamma) \quad (2.8)$$

reversing the orientation of all 1-paths.

Around every vertex $v \in V(\Gamma)$ there are three local edges e_1, e_2, e_3 or three local paths $\vec{e}(v)_1, \vec{e}(v)_2, \vec{e}(v)_3$ (local means that a priori two of them can be part of a loop) or local flags. Let P_1^v be the set of 1-paths with a vertex $v \in V(\Gamma)$. The orientation forgetting projection f sends P_1^v to $E(\Gamma)_v$.

Every path $\vec{e} \in P_1(\Gamma)$ defines two vertices

$$v_s(\vec{e}), \quad v_t(\vec{e}) \in V(\Gamma) \quad (2.9)$$

the target and the source ends of an arrow. Of course

$$e \in L(\Gamma) \implies v_s(\vec{e}) = v_t(\vec{e}).$$

but even such vertex admits two local flags $\vec{e}_s \neq \vec{e}_t$.

Definition 2.1 A section o is called a *decay orientation* if every vertex $v \in V(\Gamma)$ is a target and a source for its local edges.

More precisely, any orientation o defines a section of the projection $f: P_1^v \rightarrow E(\Gamma)_v$ for any vertex v . This section must have two arrows with opposite orientation.

We have maps

$$\begin{aligned} v_s: P_1(\Gamma) &\rightarrow V(\Gamma), & v_t: P_1(\Gamma) &\rightarrow V(\Gamma) \\ (v_s \times v_t): P_1(\Gamma) &\rightarrow V(\Gamma) \times V(\Gamma) \end{aligned} \quad (2.10)$$

and the intersection of the image with the diagonal

$$(v_s \times v_t)(P_1(\Gamma)) \cap V(\Gamma)_\Delta = v(L(\Gamma)). \quad (2.11)$$

A path of length d in Γ is an ordered sequence $\vec{e}_1, \dots, \vec{e}_d$ of oriented edges such that for every number i

$$v_t(\vec{e}_i) = v_s(\vec{e}_{i+1}). \quad (2.12)$$

Let $P_d(\Gamma)$ be the set of paths of length d in Γ . Every path $(\vec{e}_1, \dots, \vec{e}_d) \in P_d(\Gamma)$ defines two vertices

$$v_s(\vec{e}_1, \dots, \vec{e}_d), \quad \text{and} \quad v_t(\vec{e}_1, \dots, \vec{e}_d) \in V(\Gamma) \quad (2.13)$$

the source and target of a path; this defines maps

$$\begin{aligned} v_s: P_d(\Gamma) &\rightarrow V(\Gamma), & v_t: P_d(\Gamma) &\rightarrow V(\Gamma) \\ (v_s \times v_t): P_d(\Gamma) &\rightarrow V(\Gamma) \times V(\Gamma). \end{aligned} \quad (2.14)$$

The inverse image of the intersection with the diagonal

$$L_d(\Gamma) = (v_s \times v_t)^{-1}(V(\Gamma)_\Delta) \quad (2.15)$$

is the set of loops of length d . In particular,

$$L_1(\Gamma) = L(\Gamma).$$

For a vertex $v \in V(\Gamma)$ we have the set of loops

$$L_d(\Gamma)_v = (v_s \times v_t)^{-1}(v) \quad (2.16)$$

and the union

$$L_\infty(\Gamma)_v = \bigcup_{d=1}^{\infty} L_d(\Gamma)_v \quad (2.17)$$

admits a group structure. This is the combinatorial fundamental group

$$\pi_1^C(\Gamma)_v = L_\infty(\Gamma)_v \quad (2.18)$$

of our graph Γ . It is easy to check

Proposition 2.2 (1) *The combinatorial fundamental group $\pi_1^C(\Gamma, v_0)$ is equal to the usual fundamental group $\pi_1(\Gamma, v_0)$,*

(2) *the usual fundamental group $\pi_1(\Gamma)$ is the free group with g generators, where g is genus of Γ .*

Indeed, contracting $2g - 3$ simple edges of Γ gives a bouquet of g circles.

Now consider the \mathbb{Z} -module $V^\mathbb{Z}$ of all formal linear combinations of vertices of $V(\Gamma)$ with coefficients in \mathbb{Z} . Then any graph Γ of genus g can be represented as an endomorphism (matrix)

$$q_\Gamma: V^\mathbb{Z} \rightarrow V^\mathbb{Z} \quad (2.19)$$

with coefficient α_{v_i, v_j} = the number of edges joining v_i and v_j . Thus every graph Γ defines an integer quadratic form q_Γ on the \mathbb{Z} -module $V^\mathbb{Z}$. For example,

$$\text{for the genus 2 loop free graph } \Theta: \quad q_\Theta = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \quad (2.20)$$

$$\text{for the 2-loop graph of genus 2:} \quad q_{\Gamma_2} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (2.21)$$

$$\text{for the genus 3 multi-theta graph:} \quad q_{\Theta_3} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad (2.22)$$

(see Figure 1, p. 18). All the coefficients are nonnegative, and for a trivalent graph, the sum of the coefficients along every row and column is equal to 3.

The permutation group S_{2g-2} acts on V_{2g-2} by renumbering the vertices, and transforms matrices of the quadratic forms.

Definition 2.3 As usual, a graph Γ is *hyperbolic* if there are two subsets $V_+, V_- \subset V_{2g-2}$ such that the subspaces $V_{\pm}^{\mathbb{Z}}$ are isotropic with respect to q_{Γ} .

For such a graph the matrix q_{Γ} has the block form

$$q_{\Gamma} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \quad (2.23)$$

where the blocks

$$\frac{1}{2}q_{\Gamma} = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \in \text{Hom}(V_+^{\mathbb{Z}}, V_-^{\mathbb{Z}}) \quad (2.24)$$

satisfy the same conditions on rows and columns as before. In this case the set of edges $E(\Gamma)$ can be presented as a collection of triples

$$E(\Gamma) = \bigcup_{v \in V_+} E(\Gamma)_v \quad (2.25)$$

and we have the map

$$v_+ : E(\Gamma) \rightarrow V_+ \quad (2.26)$$

which defines an orientation

$$o_{V_+} : E(\Gamma) \rightarrow P_1(\Gamma). \quad (2.27)$$

If our graph Γ is connected then this is a decay orientation (see Definition 3.1).

Recall that the \mathbb{Z} -module of formal linear combinations over \mathbb{Z} of trivalent graphs is an algebra with respect to the multiplication induced by disjoint union of graphs. For example,

$$q_{\Theta^2} = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \quad (2.28)$$

and so on.

3 SU(2)-gauge theory on trivalent graphs

3.1 Space of connections

A connection a on the trivial SU(2)-bundle on Γ is a map

$$a: P_1(\Gamma) \rightarrow \text{SU}(2) \quad (3.1)$$

such that for the involution (2.5) we have

$$a(i_e(\vec{e})) = a(\vec{e})^{-1}. \quad (3.2)$$

Then the “path integral” is given by

$$a(\vec{e}_1, \dots, \vec{e}_d) = a(\vec{e}_1) \cdots a(\vec{e}_d) \in \text{SU}(2), \quad (3.3)$$

that is, it defines a map

$$a: P_d(\Gamma) \rightarrow \text{SU}(2) \quad (3.4)$$

such that for the orientation reversing involution i_Δ we have

$$a(i(\vec{e}_1, \dots, \vec{e}_d)) = a(\vec{e}_1, \dots, \vec{e}_d)^{-1}. \quad (3.5)$$

In the same vein we have the monodromy map for loops

$$a: L_d(\Gamma) \rightarrow \text{SU}(2). \quad (3.6)$$

Obviously every connection is *flat*.

Let $\mathcal{A}(\Gamma)$ be the space of connections, that is, the space of functions (3.1) subject to the constraint (3.2):

$$\mathcal{A}(\Gamma) = \{a \in \text{SU}(2)^{P_1(\Gamma)} \mid a(i_e(\vec{e})) = a(\vec{e})^{-1}\}. \quad (3.7)$$

3.2 Gauge transformations group

Every element \tilde{g} of the gauge transformations group $\mathcal{G}(\Gamma)$ is a function

$$\tilde{g}: V(\Gamma) \rightarrow \text{SU}(2), \quad (3.8)$$

that is,

$$\mathcal{G}(\Gamma) = \text{SU}(2)^{V(\Gamma)} \quad (3.9)$$

with componentwise multiplication. This group acts on the space of connections $\mathcal{A}(\Gamma)$ by the rule

$$\tilde{g}(a(\vec{e})) = \tilde{g}(v_s(\vec{e})) \cdot a(\vec{e}) \cdot \tilde{g}(v_t(i(\vec{e}))). \quad (3.10)$$

Recall that $\tilde{g}(v_t(i(\vec{e}))) = \tilde{g}(v_t(\vec{e}))^{-1}$.

The space of gauge orbits

$$\mathcal{B}(\Gamma) = \mathcal{A}(\Gamma)/\mathcal{G}(\Gamma) \quad (3.11)$$

is the space of representation classes of the fundamental group of Γ , as expected, because all our connections are flat:

$$\mathcal{A}(\Gamma)/\mathcal{G}(\Gamma) = \text{CLRep}(\pi_1(\Gamma)), \quad (3.12)$$

where as usual for a group G

$$\text{Rep}(G) = \text{Hom}(G, \text{SU}(2)) \quad (3.13)$$

is the space of representations and the quotient by the adjoint action

$$\text{CLRep}(G) = \text{Hom}(G, \text{SU}(2))/\text{Ad SU}(2) \quad (3.14)$$

is the space of representation classes.

Now the gauge transformation group $\mathcal{G}(\Gamma)$ contains the diagonal subgroup

$$\text{SU}(2)_\Delta \subset \mathcal{G}(\Gamma) \quad (3.15)$$

of constant functions (3.8). The action of this subgroup defines the space of constant orbits

$$\text{CL } \mathcal{A}(\Gamma) = \mathcal{A}(\Gamma)/\text{SU}(2)_\Delta. \quad (3.16)$$

For every vertex $v \in V(\Gamma)$ we have the subgroup

$$\mathcal{G}(\Gamma)_v = \{\tilde{g} \in \mathcal{G}(\Gamma) \mid \tilde{g}(v) = \text{id}\} \quad (3.17)$$

of gauge transformations preserving the framing at v . This is a normal subgroup, and

$$\mathcal{G}(\Gamma)/\mathcal{G}(\Gamma)_v = \text{SU}(2)_\Delta. \quad (3.18)$$

Thus the full group of gauge transformations is the semidirect product of $\mathcal{G}(\Gamma)_v$ and $\mathrm{SU}(2)_\Delta$.

The quotient

$$\mathcal{A}(\Gamma)/\mathcal{G}(\Gamma)_v = \mathrm{Rep}(\pi_1(\Gamma, v)) \quad (3.19)$$

is the orbit space of connections framed at v ; this space depends on the choice of v .

Thus the quotient map (3.12)

$$P: \mathcal{A}(\Gamma) \rightarrow \mathrm{CLRep}(\pi_1(\Gamma)) \quad (3.20)$$

can be decomposed as follows

$$\mathcal{A}(\Gamma) \xrightarrow{P} \mathrm{Rep}(\pi_1(\Gamma)) \xrightarrow{/\mathrm{SU}(2)_\Delta} \mathrm{CLRep}(\pi_1(\Gamma)) \quad (3.21)$$

or

$$\mathcal{A}(\Gamma) \xrightarrow{/\mathrm{SU}(2)_\Delta} \mathrm{CL}\mathcal{A}(\Gamma) \xrightarrow{P_{cl}} \mathrm{CLRep}(\pi_1(\Gamma)). \quad (3.22)$$

The involution i_Δ acts on the space $\mathcal{A}(\Gamma)$ of connections

$$i_\Delta^*: \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma) \quad (3.23)$$

and there exists an element

$$\tilde{g}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2)_\Delta \subset \mathcal{G}(\Gamma) \quad (3.24)$$

such that

$$i_\Delta^* = \tilde{g}_i. \quad (3.25)$$

(Here $i = \sqrt{-1}$.) Thus the involution i_Δ^* (3.23) acts trivially on $\mathrm{CL}\mathcal{A}(\Gamma)$. Recall that a gauge fixing is a section of the projection P (3.20).

Definition 3.1 A partial gauge fixing is a section

$$s: \mathrm{CLRep}(\pi_1(\Gamma)) \rightarrow \mathrm{CL}\mathcal{A}(\Gamma).$$

3.3 The map conj

Recall that the set $\text{Conj}(\text{SU}(2))$ of conjugacy classes of elements of $\text{SU}(2)$ can be described as the interval $[0, 1]$ with the map

$$\text{conj}: \text{SU}(2) \rightarrow \text{Conj}(\text{SU}(2)) = [0, 1] \quad (3.26)$$

that sends a matrix $g \in \text{SU}(2)$ to

$$\text{conj}g = \frac{1}{\pi} \cdot \cos^{-1} \left(\frac{1}{2} \text{Tr } g \right) \in [0, 1]. \quad (3.27)$$

Using this map coordinatewise gives the map

$$\text{conj}: \text{CL } \mathcal{A}(\Gamma) \rightarrow [0, 1]^{3g-3} = \prod_{e \in E(\Gamma)} [0, 1]_e, \quad (3.28)$$

which is obviously surjective; it is the composite

$$\mathcal{A}(\Gamma) \xrightarrow{\text{SU}(2)_\Delta} \text{CL } \mathcal{A}(\Gamma) \rightarrow [0, 1]^{3g-3} = \prod_{e \in E(\Gamma)} [0, 1]_e, \quad (3.29)$$

and the involution i_Δ preserves its fibers.

3.4 Abelian gauge theory

Abelian gauge theory on Γ is a good model for non-Abelian gauge theory. A $\text{U}(1)$ -connection a is a map

$$a: P_1(\Gamma) \rightarrow \text{U}(1) \quad (3.30)$$

such that

$$a(i_e(\vec{e})) = a(\vec{e})^{-1}. \quad (3.31)$$

The “path integral” is given by

$$a(\vec{e}_1, \dots, \vec{e}_d) = \prod_{\vec{e} \in P_1(\Gamma)} a(\vec{e}) \in \text{U}(1), \quad (3.32)$$

and so on. Let

$$\mathcal{A}_{\text{U}(1)}(\Gamma) = \{a \in \text{U}(1)^{P_1(\Gamma)} \mid a(i_e(\vec{e})) = a(\vec{e})^{-1}\}. \quad (3.33)$$

be the space of $U(1)$ -connections. Then we have the same involution

$$\mathcal{A}_{U(1)}(\Gamma) \xrightarrow{i_e^*} \mathcal{A}_{U(1)}(\Gamma). \quad (3.34)$$

Every element \tilde{u} of the gauge transformations group $\mathcal{G}_{U(1)}(\Gamma)$ is a function

$$\tilde{u}: V(\Gamma) \rightarrow U(1). \quad (3.35)$$

This group acts on the space of connections $\mathcal{A}_{U(1)}(\Gamma)$ by the same rule:

$$\tilde{u}(a(\vec{e})) = \tilde{u}(v_s(\vec{e})) \cdot a(\vec{e}) \cdot \tilde{u}(v_t(i(\vec{e}))). \quad (3.36)$$

There is however one important difference between the Abelian and non-Abelian theories. Namely the diagonal group

$$U(1)_\Delta \subset \mathcal{G}_{U(1)}(\Gamma) \quad (3.37)$$

acts trivially. Thus

$$\mathcal{B}_{U(1)}(\Gamma) = \mathcal{A}_{U(1)}(\Gamma) / \mathcal{G}_{U(1)}(\Gamma) = \text{Hom}(\pi_1(\Gamma), U(1)) = U(1)^g. \quad (3.38)$$

Let

$$P_a: \mathcal{A}_{U(1)}(\Gamma) \rightarrow U(1)^g \quad (3.39)$$

be the projection map.

The Abelian and non-Abelian theories are related by the following map:

$$d: \mathcal{A}_{U(1)}(\Gamma) \rightarrow \mathcal{A}(\Gamma) \quad \text{given by} \quad d(a(\vec{e})) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}. \quad (3.40)$$

d is equivariant with respect to every involution i_e^* . Obviously the image $d(\mathcal{A}_{U(1)}(\Gamma)) \subset \mathcal{A}(\Gamma)$ is a 2-section of the projection conj , that is, the composite

$$\text{conj} \circ d: \mathcal{A}_{U(1)}(\Gamma) \rightarrow \prod_{e \in E(\Gamma)} [0, 1]_e \quad (3.41)$$

is the factorization by the involution i_Δ^* .

It is easy to check

Proposition 3.2 *In the chain of maps*

$$\mathcal{A}_{U(1)}(\Gamma) \xrightarrow{d} \mathcal{A}(\Gamma) \xrightarrow{P} U(1)^g \in \text{CLRep}(\pi_1(\Gamma)) \quad (3.42)$$

the composite $d \circ P$ is the projection map P_a (3.39).

4 From trivalent graph to handlebody

4.1 Pumping

Any trivalent graph Γ defines a handlebody $\tilde{\Gamma}$, that is, a 3-manifold with boundary $\partial\tilde{\Gamma} = \Sigma_\Gamma$ by the “pumping trick” (see [6]): pump up the edges of Γ to tubes and the vertices to small 2-spheres. This gives a Riemann surface Σ_Γ of genus g with a tube \tilde{e} for every $e \in E(\Gamma)$ and a *trinion* \tilde{v} for every $v \in V(\Gamma)$, where each trinion is a 2-sphere with three disjoint holes. The isotopy classes of meridian circles of tubes define $3g - 3$ disjoint, noncontractible, pairwise nonisotopic circles $\{C_e\}, e \in E(\Gamma)$ on Σ . The complement is the union

$$\Sigma_\Gamma \setminus \bigcup_{e \in E(\Gamma)} C_e = \bigcup_{i=1}^{2g-2} \tilde{v}_i \quad (4.1)$$

of $2g - 2$ trinions (pairs of pants) corresponding to vertices of our graph Γ . Certainly, every trivalent graph Γ is geometrically equivalent to the handlebody $\tilde{\Gamma}$ or to the Riemann surface Σ_Γ with fixed pants decomposition (4.1). Moreover, we have the identification

$$\pi_1(\tilde{\Gamma}) = \pi_1(\Gamma) \quad (4.2)$$

and the epimorphism

$$r: \pi_1(\Sigma_\Gamma) \rightarrow \pi_1(\tilde{\Gamma}) = \pi_1(\Gamma). \quad (4.3)$$

We can consider our graph Γ as a skeleton inside the handlebody $\tilde{\Gamma}$. Then for every edge e inside the full tube with the boundary \tilde{e} there exists a disc Δ in the handlebody with the boundary on \tilde{e} meeting the edge e transversely at one point such that

$$\partial\Delta = C_e, \quad (4.4)$$

and we can move Γ to the boundary of the handlebody to get an embedding

$$j: \Gamma \rightarrow \Sigma_\Gamma \quad (4.5)$$

defined up to isotopy.

Now fix a point $p_e \in C_e$ on each circle C_e and a vertex $v_0 \in V(\Gamma)$. We identify it with the point $j(v_0) \in \Sigma_\Gamma$. Then we can view (4.3) as being the epimorphism of the exact sequence

$$1 \rightarrow \ker r \rightarrow \pi_1(\Sigma_\Gamma, v_0) \xrightarrow{r} \pi_1(\Gamma, v_0) \rightarrow 1 \quad (4.6)$$

if we fix paths from v_0 to every point p_e .

The orientation of every path of length 1 $\vec{e} \in P_1(\Gamma)$ and the pumping normal vector field define an orientation of the tube \tilde{e} and hence an orientation of the meridian C_e of (4.1). Thus we get the cycle $C_{\vec{e}}$ joined by the path with the point v . This gives a collection of homotopy classes considered as elements of the fundamental group $\pi_1(\Sigma_\Gamma, v_0)$

$$\{C_{\vec{e}} \mid e \in E(\Gamma)\} \subset \pi_1(\Sigma_\Gamma, v_0). \quad (4.7)$$

More precisely,

$$\{C_{\vec{e}}\} \subset \ker r. \quad (4.8)$$

Now sending \vec{e} to $C_{\vec{e}}$ and a loop of length d in Γ given by a sequence $\vec{e}_1, \dots, \vec{e}_d$ to

$$p(\vec{e}_1, \dots, \vec{e}_d) = C_{\vec{e}_d} \circ \dots \circ C_{\vec{e}_1} \in \pi_1(\Sigma_\Gamma, v_0) \quad (4.9)$$

gives a map of the collection of loops through v_0 to $\pi_1(\Sigma_\Gamma, v_0)$. Hence we get the homomorphism

$$q: \pi_1(\Gamma, v_0) \rightarrow \ker r \subset \pi_1(\Sigma_\Gamma, v_0). \quad (4.10)$$

The composite $r \circ q$ defines the “complex” of groups

$$\begin{aligned} \dots \xrightarrow{r \circ q} \pi_1(\Sigma_\Gamma, v_0) \xrightarrow{r \circ q} \pi_1(\Sigma_\Gamma, v_0) \xrightarrow{r \circ q} \dots \\ (r \circ q)^2 = \text{id} \end{aligned} \quad (4.11)$$

with cohomology

$$\ker(r \circ q) / \text{im}(r \circ q) = \ker r / \text{im } q = H(\Gamma). \quad (4.12)$$

4.2 From gauge theory on Σ_Γ to gauge theory on Γ

Let $\mathcal{A}_F(\Sigma_\Gamma)$ be the space of $\mathrm{SU}(2)$ -connections on the trivial vector bundle of rank 2 on Σ_Γ , \mathcal{G} the corresponding gauge group and

$$\mathrm{Rep}(\pi_1(\Sigma_\Gamma, v_0)) = \mathcal{A}_F(\Sigma_\Gamma)/\mathcal{G}_{v_0} \quad (4.13)$$

the framed gauge-orbit space of flat connections.

The main construction relating gauge theories on a Riemann surface Σ_Γ and on Γ is the map

$$\begin{aligned} m: \mathrm{Rep}(\pi_1(\Sigma_\Gamma, v_0)) &\rightarrow \mathcal{A}(\Gamma) \\ m(\rho)(\vec{e}) &= \rho(C_{\vec{e}}). \end{aligned} \quad (4.14)$$

$\mathrm{SU}(2)$ acts on the target space of this map by the adjoint action and on the source as the diagonal subgroup $\mathrm{SU}(2)_\Delta$ under the action of the gauge group \mathcal{G} . This map is equivariant with respect to these actions, so we can factorize it as a map of quotients

$$m: \mathrm{CLRep}(\pi_1(\Sigma_\Gamma)) \rightarrow \mathrm{CL} \mathcal{A}(\Gamma) \quad (4.15)$$

(see (3.16)). For any sequence e_1, e_2, \dots, e_n of edges, we construct the map

$$m_{(e_1, e_2, \dots, e_n)} = \prod_{i=1}^n i_{e_i} \circ m, \quad (4.16)$$

where i_{e_i} are involutions (2.4) and (3.2).

Different composites of this map with maps from gauge theory on Γ provide important fibrations of $\mathrm{CLRep}(\pi_1(\Sigma_\Gamma))$. The first of these is the composite

$$\mathrm{conj} \circ m = \pi_\Gamma: \mathrm{CLRep}(\pi_1(\Sigma_\Gamma)) \rightarrow \Delta_\Gamma \subset \prod_{e \in E(\Gamma)} [0, 1]_e, \quad (4.17)$$

(see (3.28)) which is nothing other than the real polarization (see [7], [8] and below). This fibration is given by the moment map of $(3g - 3)$ -torus action and is described perfectly.

The second fibration is given by the quotient projection

$$P \circ m: \mathrm{Rep}(\pi_1(\Sigma_\Gamma)) \rightarrow \mathrm{Rep}(\pi_1(\Gamma)). \quad (4.18)$$

From the direct description of maps it is easy to prove the following

Proposition 4.1 *This map is nothing other than the composite*

$$\mathrm{Rep}(\pi_1(\Sigma_\Gamma, v_0)) \xrightarrow{\mathrm{res}} \mathrm{Rep}(\ker r, v_0) \xrightarrow{q^*} \mathrm{Rep}(\pi_1(\Gamma)), v_0) \quad (4.19)$$

where the first map is the restriction map to a subgroup and the second is induced by the group homomorphism q (4.10).

Recall that $\mathrm{CLRep}(\pi_1(\Gamma))$ is called the unitary Schottky space of genus g and denoted by the symbol uS_g [6]. This space can be presented as a homogeneous space

$$uS_g = \mathrm{SU}(2)^g / \mathrm{Ad}_{\mathrm{diag}} \mathrm{SU}(2) \quad (4.20)$$

where $\mathrm{Ad}_{\mathrm{diag}} \mathrm{SU}(2)$ is the diagonal adjoint action on the direct product (see [6]). The map r^* (4.3) embeds the unitary Schottky space

$$r^*: uS_g \hookrightarrow \mathrm{CLRep}(\pi_1(\Sigma_\Gamma)) \quad (4.21)$$

but on the other hand the composite (4.18) gives the surjective map

$$P_{cl} \circ m: \mathrm{CLRep}(\pi_1(\Sigma_\Gamma)) \rightarrow uS_g. \quad (4.22)$$

Thus we have the map

$$P_{cl} \circ m \circ r^*: \mathrm{CLRep}(\pi_1(\Sigma_\Gamma)) \rightarrow \mathrm{CLRep}(\pi_1(\Sigma_\Gamma)) \quad (4.23)$$

inducing the cohomology homomorphism

$$(P_{cl} \circ m \circ r^*)^*: H^*(\mathrm{CLRep}(\pi_1(\Sigma_\Gamma)), \mathbb{Z}) \rightarrow H^*(\mathrm{CLRep}(\pi_1(\Sigma_\Gamma)), \mathbb{Z}). \quad (4.24)$$

The Florentino conjecture compares the target spaces of two projections $P \circ m$ and $\mathrm{conj} \circ m = \pi_\Gamma$, that is, the spaces uS_g and Δ_Γ .

Conjecture 4.2 (C. Florentino) *There exists a one-to-one map*

$$f: \Delta_\Gamma \rightarrow uS_g$$

which is smooth inside Δ_Γ and on the nonsingular part of uS_g and continuous everywhere.

In the genus 2 case

$$\Delta_\Theta = \mathrm{CLRep}(\pi_1(\tilde{v}_i)) = uS_2 \quad (4.25)$$

(see [7, Section 3]). In what follows, we

- (1) give uS_g the structure of a polytope (more precisely, the direct product of $g - 1$ copies of 3-dimensional tetrahedrons) and
- (2) construct the natural polytope embedding Δ_Γ to uS_g .

4.3 From gauge theory on Σ_Γ to gauge theory on a trinion

Recall from (3.16) that $\text{CL } \mathcal{A}(\Gamma) = \mathcal{A}(\Gamma)/\text{SU}(2)_\Delta$. For every trinion \tilde{v} we have the map

$$m_v: \text{CLRep}(\pi_1(\tilde{v})) \rightarrow \text{CL } \mathcal{A}(\Gamma). \quad (4.26)$$

$$m_v(\rho)(\vec{e}) = \rho(C_{\vec{e}}).$$

for $e \in E(\Gamma)_v$.

Every vertex v has three local edges $(e_1, e_2, e_3) = E(\Gamma)_v$ which define the 3-cube

$$[0, 1]_v^3 = \prod_{e \in E(\Gamma)_v} [0, 1]_e. \quad (4.27)$$

Proposition 4.3 (1) *The image of the composite*

$$m_v \circ \text{conj}: \text{CLRep}(\pi_1(\tilde{v})) \rightarrow [0, 1]^{3g-3} = \prod_{e \in E(\Gamma)} [0, 1]_e$$

is contained in $[0, 1]_v^3 = \prod_{e \in E(\Gamma)_v} [0, 1]_e$.

(2) *The image $m(\text{CLRep}(\pi_1(\tilde{v})))$ is a section of the projection conj* (3.28)

$$m_v \circ \text{conj}: \text{CLRep}(\pi_1(\tilde{v})) \rightarrow \prod_{v \in E(\Gamma)_v} [0, 1]_e. \quad (4.28)$$

(3) *The image $\text{conj}(\text{CLRep}(\pi_1(\tilde{v})))$ is the tetrahedron Δ_v inscribed in the cube given by triangle inequalities*

$$|t_1 - t_2| \leq t_3 \leq t_1 + t_2$$

where t_i is a coordinate on the interval $[0, 1]_{e_i}$ for $e_i \in E(\Gamma)_v$. Hence

(4)

$$\text{CLRep}(\pi_1(\tilde{v})) = \Delta_v = \Delta_\Theta = uS_2.$$

(for the proof see [7, Proposition 3.1]).

Now for every $v \in V(\Gamma)$ and its corresponding trinion \tilde{v} we have the restriction map

$$\text{res}_v: \text{CLRep}(\pi_1(\Sigma_\Gamma)) \rightarrow \text{CLRep}(\pi_1(\tilde{v})). \quad (4.29)$$

This map is induced by the natural map

$$r_v: \pi_1(\tilde{v}) \rightarrow \pi_1(\Sigma_\Gamma). \quad (4.30)$$

The group $\pi_1(\tilde{v})$ is the free group with 2 generators. Let

$$I_v \subset \pi_1(\Sigma_\Gamma) \quad (4.31)$$

be the image of r_v . Then

$$\text{im res}_v = \text{CLRep}(I_v) \subset \text{CLRep}(\pi_1(\tilde{v})) = \Delta_v \quad (4.32)$$

(see Proposition 4.3, (4)). For example, if $v \in v(L(\Gamma))$ is a vertex on a loop $e \in L(\Gamma)$ then $I_v = \mathbb{Z}$ and $\text{im res}_v = [0, 1]_e$.

4.4 Half Riemann surface Σ_Γ

We now give the hyperbolic graph (2.23) a one-to-one map

$$h: V_+ \rightarrow V_-. \quad (4.33)$$

Then as in Figure 2, p. 18, we have:

(1)

$$V_+ \cap h(V_+) = \emptyset, \quad (V_+ \cup V_-) = V(\Gamma), \quad |V_\pm| = g - 1;$$

(2) for every pair of distinct vertices $v_i, v_j \in V_+$,

$$E(\Gamma)_{v_i} \cap E(\Gamma)_{v_j} = \emptyset.$$

Definition 4.4 The union

$$\Sigma_\Gamma^+ = \bigcup_{v \in V_+} \tilde{v}$$

is called a half of the Riemann surface Σ_Γ .

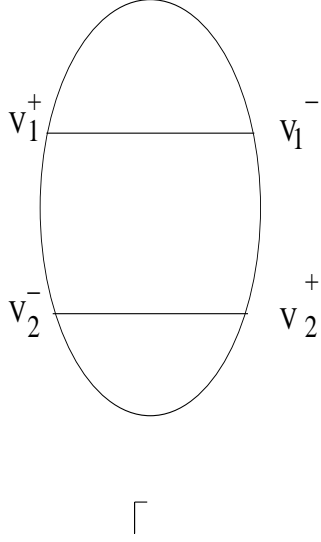


Figure 1

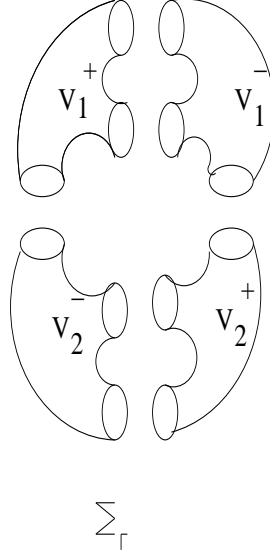


Figure 2

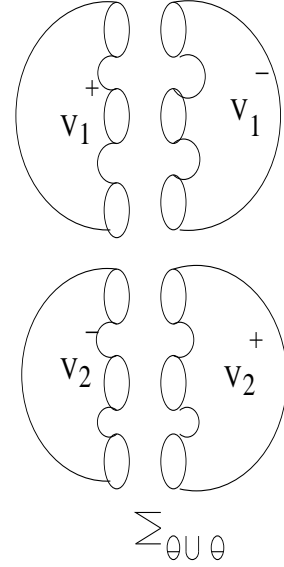


Figure 3

Thus a half of the Riemann surface Σ_Γ^+ is the disjoint union of trinions and it has a mirror $\Sigma_\Gamma^- = h(\Sigma_\Gamma^+)$ such that

$$\Sigma_\Gamma^+ \cup \Sigma_\Gamma^- = \Sigma_\Gamma. \quad (4.34)$$

A decomposition of this type is an analogue of a Heegard decomposition in the 2-dimensional case and we can apply to this situation all the constructions of [1, Seminar 4] (see “Verlinde property”). The two halves Σ_Γ^+ and Σ_Γ^- are completely symmetric, and gluing is given by the matrix q_Γ (see the end of section 2). Thus this matrix plays the role of an element of the modular group in the 3-dimensional case.

Now the fundamental group of Σ_Γ^+ is given by

$$\pi_1(\Sigma_\Gamma^+) = \prod_{v \in V_+} \pi_1(\tilde{v}), \quad (4.35)$$

and hence

$$\text{CLRep}(\pi_1(\Sigma_\Gamma^+)) = \prod_{v \in V_+} \text{CLRep}(\pi_1(\tilde{v})). \quad (4.36)$$

Thus we have a natural homomorphism

$$r_+(\Gamma) : \pi_1(\Sigma_\Gamma^+) \rightarrow \pi_1(\Sigma_\Gamma) \quad (4.37)$$

and the induced map

$$r_+^*(\Gamma) : \text{CLRep}(\pi_1(\Sigma_\Gamma)) \rightarrow \text{CLRep}(\pi_1(\Sigma_\Gamma^+)) = \prod_{v \in V_+} \text{CLRep}(\pi_1(\tilde{v})). \quad (4.38)$$

The boundary of our half of the Riemann surface Σ_Γ^+

$$\partial \Sigma_\Gamma^+ = \{C_e\}, \quad e \in E(\Gamma) \quad (4.39)$$

is the disjoint union of cycles (4.1) with the natural orientation coming from the orientations of trinions.

4.5 Half Riemann surface $\Sigma_{\Theta^{g-1}}$

For genus 2 there is a unique connected trivalent graph without loops Θ (denoted by this symbol for the shape). This graph gives the disconnected graph of genus g

$$\Theta^{g-1} = \Theta \cup \dots \cup \Theta \quad (4.40)$$

which is disjoint union of $g - 1$ copies of Θ .

In particular the fundamental group

$$\pi_1(\Theta^{g-1}) = \prod_{g-1 \text{ copies}} \pi_1(\Theta) \quad (4.41)$$

We can apply all the constructions of Section 3.1 to define the space of connections $\mathcal{A}(\Theta^{g-1})$, the gauge group $\mathcal{G}(\Theta^{g-1})$ action and the gauge orbit space of Section 3.3

$$\mathcal{A}(\Theta^{g-1})/\mathcal{G}(\Theta^{g-1}) = \text{CLRep}(\pi_1(\Theta^{g-1})) \quad (4.42)$$

and the map conj of Section 3.4.

Moreover pumping Θ we get the compact Riemann surface Σ_Θ of genus 2 with the trinion decomposition of Figure 3. Thus the Riemann surface $\Sigma_{\Theta^{g-1}}$ is the disjoint union

$$\Sigma_{\Theta^{g-1}} = \Sigma_\Theta \cup \dots \cup \Sigma_\Theta \quad (4.43)$$

of $g-1$ copies of a Riemann surface of genus 2 with the trinion decomposition (4.1).

A small difference between the case of connected and disjoint curves is that we can't fix a basic point v_0 . Thus we have to consider spaces of representation classes. We have the map (4.15)

$$m: \text{CLRep}(\pi_1(\Sigma_{\Theta^{g-1}})) \rightarrow \text{CL } \mathcal{A}(\Theta^{g-1}). \quad (4.44)$$

and the composite (4.7)

$$\text{conj} \circ m = \pi_{\Theta^{g-1}}: \text{CLRep}(\pi_1(\Sigma_{\Theta^{g-1}})) \rightarrow \Delta_{\Theta^{g-1}} \subset \prod_{e \in E(\Theta^{g-1})} [0, 1]_e. \quad (4.45)$$

Now consider the decomposition of Σ_Γ as an union of disjoint trinion as a many piece jigsaw puzzle (as in Figure 2). We can transform this collection to the union of disjoint trinions of a decomposition of $\Sigma_{\Theta^{g-1}}$ (as in Figure 3). Thus we may consider the half of the Riemann surface Σ_Γ as the half of the Riemann surface $\Sigma_{\Theta^{g-1}}$ with the decomposition

$$V_+ \cap h(V_+) = \emptyset, \quad V_+ \cup h(V_+) = V(\Theta^{g-1}), \quad |V_\pm| = g-1.$$

Thus the union Σ_Γ^+ is a half of two curves

$$\Sigma_\Gamma \supset \Sigma_\Gamma^+ \subset \Sigma_{\Theta^{g-1}} \quad (4.46)$$

and together with the homomorphism $r_+(\Gamma)$ (4.38), we get another

$$r_+(\Theta^{g-1}): \pi_1(\Sigma_\Gamma^+) = \pi_1(\Sigma_{\Theta^{g-1}}^+) \rightarrow \pi_1(\Sigma_{\Theta^{g-1}}) \quad (4.47)$$

which componentwise is the standard map for Riemann surfaces of genus 2.

5 Gauge quotient map

5.1 The section of conj

We have identifications

$$E(\Gamma) = E(\Theta^{g-1}), \quad V(\Gamma) = V(\Theta^{g-1}), \quad P_1(\Gamma) = P_1(\Theta^{g-1})$$

and hence

$$\mathcal{A}(\Gamma) = \mathcal{A}(\Theta^{g-1}) \quad \text{and} \quad \text{CL } \mathcal{A}(\Gamma) = \text{CL } \mathcal{A}(\Theta^{g-1}). \quad (5.1)$$

But the gauge groups $\mathcal{G}(\Gamma)$ and $\mathcal{G}(\Theta^{g-1})$ actions are entirely different. In particular, the gauge quotient map for the gauge group $\mathcal{G}(\Theta^{g-1})$ can be decomposed as follows

$$\mathrm{CL}\mathcal{A}(\Gamma) \xrightarrow{p_{\Theta}^{g-1}} \prod_{(g-1) \text{ copies}} \mathrm{CL}\mathcal{A}(\Theta) \xrightarrow{(g-1) \times p_{\Theta}} \prod_{(g-1) \text{ copies}} \mathrm{CLRep}(\pi_1(\Theta)). \quad (5.2)$$

All components of the latter map are the same

$$p_{\Theta}: \mathrm{CL}\mathcal{A}(\Theta) \rightarrow \mathrm{CLRep}(\pi_1(\Theta)). \quad (5.3)$$

We study this map later. The maps conj are absolutely the same, with the same target space $\prod_{e \in E(\Gamma)} [0, 1]_e$. Moreover, the product of the maps m_v (4.26) gives the map

$$m_+ = \prod_{v \in V_+} m_v: \prod_{v \in V_+} \mathrm{CLRep}(\pi_1(\tilde{v})) \rightarrow \mathrm{CL}\mathcal{A}(\Gamma). \quad (5.4)$$

Let

$$W = m_+ \left(\prod_{v \in V_+} \mathrm{CLRep}(\pi_1(\tilde{v})) \right) \subset \mathrm{CL}\mathcal{A}(\Gamma) \quad (5.5)$$

be the image of (5.4). A direct interpretation of the maps in the composite

$$\mathrm{CLRep}(\pi_1(\Sigma_{\Gamma})) \xrightarrow{r_+^*(\Gamma)} \prod_{v \in V_+} \mathrm{CLRep}(\pi_1(\tilde{v})) \xrightarrow{m_+} \mathrm{CL}\mathcal{A}(\Gamma), \quad (5.6)$$

gives

Proposition 5.1 (1) *The image*

$$r_+^*(\Gamma) \circ m_+(\mathrm{CLRep}(\pi_1(\Sigma_{\Gamma}))) \subset W.$$

(2) *The map (4.17)*

$$\mathrm{conj} \circ m: \mathrm{CLRep}(\pi_1(\Sigma_{\Gamma})) \rightarrow \Delta_{\Gamma} \subset \prod_{e \in E(\Gamma)} [0, 1]_e$$

is the composite of $r_+^*(\Gamma)$ and

$$\mathrm{conj} \circ m_+: \prod_{v \in V_+} \mathrm{CLRep}(\pi_1(\tilde{v})) \rightarrow \prod_{v \in V_+} \Delta_v \subset \prod_{e \in E(\Gamma)} [0, 1]_e. \quad (5.7)$$

Thus

(3)

$$\Delta_\Gamma \subset \prod_{v \in V_+} \Delta_v \subset \prod_{e \in E(\Gamma)} [0, 1]_e. \quad (5.8)$$

Proposition 5.2 *The image W is a section of the projection conj . That is, the restriction of conj*

$$\text{conj}|_W : W \rightarrow \prod_{v \in V_+} \Delta_v \subset \prod_{e \in E(\Gamma)} [0, 1]_e$$

is one-to-one.

To prove this statement, we may use the conj -map for $\Theta^{g^{-1}}$. Recall that trinions of components of the direct product W don't have boundaries in common. Thus coordinates of the cube are divided in triples corresponding vertices $v \in V_+(\Gamma) = V_+(\Theta^{g^{-1}})$. Then Proposition 4.2, (3) implies the required statement componentwise.

Corollary 5.1 *The identification $W = \prod_{v \in V_+} \Delta_v$ determines the section*

$$s : \prod_{v \in V_+} \Delta_v \rightarrow \text{CL } \mathcal{A}(\Gamma) \quad (5.9)$$

and the restriction of this section to $\Delta_\Gamma \subset \prod_{v \in V_+} \Delta_v$ determines the embedding

$$s|_{\Delta_\Gamma} : \Delta_\Gamma \rightarrow \text{CL } \mathcal{A}(\Gamma). \quad (5.10)$$

C. Florentino gave a beautiful description of the subpolytope $\Delta_\Gamma \subset \Delta_\Theta^{g^{-1}}$ in terms of symmetries of a graph Γ .

Now the restriction $s|_{\Delta_\Gamma}(\Delta_\Gamma)$ of the quotient map of $\mathcal{G}(\Gamma)$ -action sends Δ_Γ to uS_g . Hence it is just what we need to construct the natural map $f : \Delta_\Gamma \rightarrow uS_g$ for the Florentino conjecture. But the key is the restriction of the quotient map of $\mathcal{G}(\Gamma)$ -action to W , that is, comparing the gauge theory on Γ and the gauge theory on $\Theta^{g^{-1}}$:

$$s \circ P_d : W \rightarrow \text{CLRep}(\pi_1(\Gamma)), \quad (5.11)$$

where P_d is the projection of (3.22).

5.2 $\Gamma = \Theta$

In this case the handlebody $\tilde{\Theta}$ is a full pretzel. We get the Riemann surface Σ_{Θ} of genus 8 marked by tubes $\tilde{e}_1, \tilde{e}_2, \tilde{e}_6$ for every edge of the triple $E(\Gamma)$ and trinions \tilde{v}_+ and \tilde{v}_- of the pair $V(\Theta)$. The isotopy classes of meridian circles of tubes define 3 disjoint, noncontractible, pairwise nonisotopic classes c_i around e_i . The complement is the union

$$\Sigma_{\Theta} \setminus \{c_1, c_2, c_3\} = \tilde{v}_+ \cup \tilde{v}_- \quad (5.12)$$

of 2 trinions. We fix a point $p \in \tilde{v}_+$ and the natural orientation so that

$$c_1 \cdot c_2 \cdot c_3 = 1 \quad (5.13)$$

as classes of pointed loops.

Now

$$\mathcal{G}(\Theta) = \mathrm{SU}(2)^{V(\Gamma)} = \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \quad (5.14)$$

with the action

$$g_+ \cdot * \cdot g_-^{-1} \quad (5.15)$$

if $v_+ = v_s$. We finally get the map

$$P_{cl}: \mathrm{CLRep}(\pi_1(\Sigma_{\Theta})) \rightarrow \mathrm{CLRep}(\pi_1(\Theta)). \quad (5.16)$$

Proposition 5.3 *The image $i_{e_3}(W)$ is a section of the projection P_{cl} (3.22).*

The boundary components $\partial\tilde{v} = c_1, c_2, c_3$ as elements of the fundamental group of \tilde{v} constrain the equation (5.13). Consider the map

$$m_{e_3}: \mathrm{CLRep}(\pi_1(\tilde{v})) \rightarrow \mathrm{CL}\mathcal{A}(\Theta) \quad (5.17)$$

Then the component g_v of an element of the gauge group acts on these cycles as $g(\rho(c_i)) = \rho(c_i) \circ g$ for $i = 1, 2$ and $g(\rho(c_3)) = g^{-1}\rho(c_3)$. To preserve the equation (5.13) we have

$$\rho(c_1) \circ g \circ \rho(c_2) \circ g \circ g^{-1}\rho(c_3) = 1. \quad (5.18)$$

From this we have $g = 1$ and we are done.

Corollary 5.2 *This identification*

$$W = \text{CLRep}(\pi_1(\Theta)) = uS_2 \quad (5.19)$$

defines

- (1) *the polytope structure on uS_2 and*
- (2) *the polytope equivalence*

$$f: \Delta_\Theta \rightarrow uS_2. \quad (5.20)$$

Thus the Florentino conjecture is true in the case $g = 2$. We used this identification for the construction of non-Abelian theta functions in the first nontrivial case $g = 2$ considered in [10].

5.3 General case

We show how to prove the statement of Proposition 5.3 for a multi-theta graph of any genus such as Figure 3, p 18. The principle of the proof is just as before. Namely in (5.18) the element g after $\rho(c_1)$ doesn't have to be the same as after $\rho(c_2)$. Then in the same vein g has to be 1.

To apply this hint, remark that the set of vertices $V(\Gamma)$ decompose in pairs $(v_i^+, h(v_i^+) = v_i^-)$. Let us start with the pair v_1^\pm . Using the adjoint action we may suppose that $g_{v_1}^- = 1$. Assume

$$\partial \tilde{v}_1^+ \cap \partial \tilde{v}_1^- = c_2 \cup c_3 \quad (5.21)$$

and

$$c_1 = \partial \tilde{v}_1^+ \cap \partial \tilde{v}_2^-$$

We can choose the map $m_{\{e\}}$ of (4.16) so that e_1 and e_2 have opposite orientations. Then we have equations

$$c_1 \circ c_2 \circ c_3 = 1$$

and

$$\rho(c_1) \circ g_2^- \circ \rho(c_2) \circ g_1^+ \circ (g_1^+)^{-1} \circ \rho(c_3) = 1. \quad (5.22)$$

Thus $g_2^- = 1$. Applying the same construction to \tilde{v}_1^- we get $g_2^+ = 1$ also. This gives the proof for genus 3 case.

Iterating these arguments gives the proof for any hyperbolic graph of any genus.

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